THE ϕ-DIMENSION: A NEW HOMOLOGICAL MEASURE

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Abstract. In [9], K. Igusa and G. Todorov introduced two functions ϕ and ψ, which are natural and important homological measures generalising the notion of the projective dimension. These Igusa-Todorov functions have become into a powerful tool to understand better the finitistic dimension conjecture.

In this paper, for an artin R-algebra A and the Igusa-Todorov function ϕ, we characterise the ϕ-dimension of A in terms of the bifunctors $\operatorname{Ext}^i_A(-,-)$. Furthermore, by using this characterisation of the ϕ-dimension, we show that the finiteness of the ϕ-dimension of an artin algebra is invariant under derived equivalences. As an application of this result, we generalise the classical Bongartz's result [2, Corollary 1] as follows: For an artin algebra A, a tilting A-module T and the endomorphism algebra $B = \operatorname{End}_A(T)^{op}$, we have that $\phi\dim(A) - \operatorname{pd} T \leq \phi\dim(B) \leq \phi\dim(A) + \operatorname{pd} T$.

1. Introduction

In this paper we shall consider artin R-algebras and finitely generated left modules. For an artin algebra A, we denote by $\operatorname{mod}(A)$ the category of finitely generated left A-modules. Furthermore, $\operatorname{proj}(A)$ denotes the class of finitely generated projective A-modules and $\operatorname{inj}(A)$ denotes the class of finitely generated injective A-modules. Moreover, $\operatorname{pd} M$ stands for the projective dimension of any $M \in \operatorname{mod}(A)$. We recall that

$$\text{fin.dim}\,(A) := \sup\{\operatorname{pd} M \ : \ M \in \operatorname{mod}(A) \text{ and } \operatorname{pd} M < \infty\}$$

is the so-called finitistic dimension of A, and also that

$$\text{gl.dim}\,(A) := \sup\{\operatorname{pd} M \ : \ M \in \operatorname{mod}(A)\}$$

is the global dimension of A. The interes in the finitistic dimension is because of the "finitistic dimension conjecture", which is still open, and states that *the finitistic dimension of any artin algebra is finite.* This conjecture is closely related with several homological conjectures, and therefore it is a centerpiece for the development of the representation theory of artin algebras. The reader could see in [8], [14], and references therein, for the development related with the finitistic dimension conjecture.

In [9], K. Igusa and G. Todorov defined two functions, denoted by ϕ and ψ, from objects in the category $\operatorname{mod}(A)$ to the natural numbers $\mathbb{N}$. These Igusa-Todorov functions determine new homological measures generalising the notion of projective dimension. Our intention is to develop the theory for such functions. In particular,

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we try to convince the readers that these functions are natural and important homological measures. In this paper, we are dealing with the ϕ-dimension.

Following [5] and [6], we recall that the ϕ-dimension of an artin R-algebra A is

$$\phi \dim (A) := \sup \{\phi (M) \ : \ M \in \mathrm{mod}\,(A)\}.$$

Since the function ϕ is a refinement of the projective dimension (see [9]), we have that

$$\text{fin.dim}\,(A) \leq \phi \dim (A) \leq \text{gl.dim}\,(A).$$

On the other hand, F. Huard and M. Lanzilotta proved in [5] that the Igusa-Todorov functions characterise self-injective algebras. They provided, in [5], an example of an algebra A showing that the global dimension of A is not enough to determine whether A is self-injective or not; and moreover, in such example they got that $\text{fin.dim}\,(A) < \phi \dim (A) < \text{gl.dim}\,(A)$. Therefore, by taking into account the previous discussion, we establish the following "ϕ-dimension conjecture":

The ϕ-dimension of any artin algebra is finite.

Observe that, the ϕ-dimension conjecture implies the finitistic dimension conjecture; and hence it could be used as a tool to deal with the finitistic dimension conjecture.

In [12], S. Pan and C. Xi proved that the finiteness of the finitistic dimension of left coherent rings is preserved under derived equivalences. Inspired by [12] and [10], we get in Section 4 the same result for the ϕ-dimension of artin algebras (see Theorem 4.10). This result could be used in order to get a better understanding of both conjectures.

In Section 2, we introduce the necessary facts and notions needed for the developing of the paper. In Section 3, we relate the ϕ-dimension, of an artin algebra A, with the bifunctors $\mathrm{Ext}^i_A(-,-)$. In order to do that, we use the well known Auslander-Reiten formulas and Yoneda's Lemma. Here the main result is the Theorem 3.6 and it has been the main ingredient for the proof of Theorem 4.10.

In Section 4, we give the proof of the invariance (under derived equivalences) of the finiteness of the ϕ-dimensions of artin algebras. That is, we prove (see Theorem 4.10) the following: If two artin algebras A and B are derived equivalent, then $\phi \dim (A) < \infty$ if and only if $\phi \dim (B) < \infty$. More precisely, if $T^\bullet$ is a tilting complex over A, with n non-zero terms, such that $B \simeq \mathrm{End}_{D(\mathrm{mod}\,(A))}(T^\bullet)^{op}$, then $\phi \dim (A) - n \leq \phi \dim (B) \leq \phi \dim (A) + n$. In particular (see Corollary 4.11), if $T^\bullet$ is given by a tilting module $T \in \mathrm{mod}\,(A)$ then $\phi \dim (A) - \mathrm{pd}\, T \leq \phi \dim (B) \leq \phi \dim (A) + \mathrm{pd}\, T$. Observe that this corollary is a generalisation of the classical Bongartz's result [2, Corollary 1].

2. Preliminaries

Let A be an artin R-algebra. We recall that $\underline{\mathrm{mod}}\,(A)$ is the stable R-category modulo projectives, whose objects are the same as in $\mathrm{mod}\,(A)$ and the morphisms are given by $\underline{\mathrm{Hom}}_A(M,\, N) := \mathrm{Hom}_A(M,\, N)/\mathcal{P}(M,\, N)$, where $\mathcal{P}(M,\, N)$ is the R-submodule of $\mathrm{Hom}_A(M,\, N)$ consisting of the morphisms $M \to N$ factoring through objects in $\mathrm{proj}\,(A)$. Similarly, we have the stable R-category modulo injectives $\overline{\mathrm{mod}}\,(A)$ and the Auslander-Reiten translation $\tau : \underline{\mathrm{mod}}\,(A) \to \overline{\mathrm{mod}}\,(A)$, which is an R-equivalence of categories (see, for example, [1]).

We start now by recalling the definition of the Igusa-Todorov function ϕ : $\mathrm{Obj}\,(\mathrm{mod}\,(A)) \to \mathbb{N}$. Let $\mathrm{K}(A)$ denote the quotient of the free abelian group generated by the set of iso-classes $\{[M] \ : \ M \in \mathrm{mod}(A)\}$ modulo the relations: (a)

$[N]-[S]-[T]$ if $N \simeq S \oplus T$ and (b) $[P]$ if P is projective. Therefore, $\mathrm{K}(A)$ is the free abelian group generated by the iso-classes of finitely generated indecomposable non-projective A-modules.

The syzygy functor $\Omega : \underline{\mathrm{mod}}\,(A) \to \underline{\mathrm{mod}}\,(A)$ gives rise to a group homomorphism $\Omega : \mathrm{K}(A) \to \mathrm{K}(A)$, where $\Omega([M]) := [\Omega(M)]$. Let $\langle M \rangle$ denote the $\mathbb{Z}$-submodule of $\mathrm{K}(A)$ generated by the indecomposable non-projective direct summands of M. Since the rank of $\Omega(\langle M \rangle)$ is less or equal than the rank of $\langle M \rangle$, which is finite, it follows from the well ordering principle that there exists the smallest non-negative integer $\phi(M)$ such that $\Omega : \Omega^n(\langle M \rangle) \to \Omega^{n+1}(\langle M \rangle)$ is an isomorphism for all $n \geq \phi(M)$. Observe that $\phi(M)$ is always finite, whereas the projective dimension $\mathrm{pd}\, M$ could be infinite.

The main properties of the Igusa-Todorov function ϕ are summarised below.

Lemma 2.1. [9, 6] *Let A be an artin R-algebra and $M, N \in \mathrm{mod}(A)$. Then, the following statements hold.*

(a) $\phi\,(M) = \mathrm{pd}\, M$ *if* $\mathrm{pd}\, M < \infty$.
(b) $\phi\,(M) = 0$ *if* M *is indecomposable and* $\mathrm{pd}\, M = \infty$.
(c) $\phi\,(M) \leq \phi\,(N \oplus M)$.
(d) $\phi\,(M) = \phi\,(N)$ *if* $\mathrm{add}\,(M) = \mathrm{add}\,(N)$.
(e) $\phi\,(M \oplus P) = \phi\,(M)$ *for any* $P \in \mathrm{proj}\,(A)$.
(f) $\phi(M) \leq \phi(\Omega\, M) + 1$.

It follows, from the above properties, that ϕ is a good refinement of the measure "projective dimension". Indeed, for modules of finite projective dimension both homological measures coincides; and in the case of infinite projective dimension, ϕ gives a finite number as a measure.

3. ϕ-dimension and the bifunctors $\mathrm{Ext}^i_A(-,-)$

Let A be an artin R-algebra. For a given $M \in \mathrm{mod}(A)$, the projective cover of M will be denoted by $\pi_M : P_0(M) \to M$. We also denote by $\mathcal{C}_A$ the abelian category of all R-functors $F : \mathrm{mod}(A) \to \mathrm{mod}(R)$. For a given functor $F \in \mathcal{C}_A$, the isomorphism class of F will be denoted by $[F]$, e.g. $[F] := \{G \in \mathcal{C}_A \,:\, G \simeq F\}$.

Finally we denote by $\underline{\mathcal{C}}_A$ the abelian category of all R-functors $F : \underline{\mathrm{mod}}\,(A) \to \mathrm{mod}(R)$. Similarly, we introduce $\overline{\mathcal{C}}_A$ by using $\overline{\mathrm{mod}}\,(A)$ instead of $\underline{\mathrm{mod}}(A)$.

Proposition 3.1. *Let A be an artin R-algebra and $M, N \in \mathrm{mod}(A)$. Then, the following conditions are equivalent.*

(a) $\mathrm{Ext}^1_A(M,-) \simeq \mathrm{Ext}^1_A(N,-)$ *in* $\mathcal{C}_A$.
(b) $M \oplus P_0(N) \simeq N \oplus P_0(M)$ *in* $\mathrm{mod}(A)$.
(c) $M \simeq N$ *in* $\underline{\mathrm{mod}}\,(A)$.
(d) $[M] = [N]$ *in* $\mathrm{K}(A)$.

Proof.(a) $\Leftrightarrow$ (c) We have that $\mathrm{Ext}^1_A(M,-) \simeq \mathrm{Ext}^1_A(N,-)$ in $\mathcal{C}_A$ if and only if, by using Auslander-Reiten formula, the functors $D\overline{\mathrm{Hom}}_A(-,\tau M)$ and $D\overline{\mathrm{Hom}}_A(-,\tau N)$ are isomorphic in $\overline{\mathcal{C}}(A)$. Moreover, the latest isomorphism is equivalent to the existence of an isomorphism $\underline{\mathrm{Hom}}_A(-,M) \simeq \underline{\mathrm{Hom}}_A(-,N)$ in $\underline{\mathcal{C}}_A$, since the Auslander-Reiten translation $\tau : \underline{\mathrm{mod}}\,(A) \to \overline{\mathrm{mod}}\,(A)$ is an R-equivalence of categories. Finally, the fact that $\underline{\mathrm{Hom}}_A(-,M) \simeq \underline{\mathrm{Hom}}_A(-,N)$ in $\underline{\mathcal{C}}_A$, is equivalent by Yoneda's Lemma to the existence of an isomorphism $M \simeq N$ in $\underline{\mathrm{mod}}\,(A)$.

(c) $\Rightarrow$ (b) Let $M \simeq N$ in $\underline{\mathrm{mod}}\,(A)$. Then we have the following diagram in $\mathrm{mod}\,(A)$

$$\begin{tikzcd} & & P_0(N) \arrow[dr, "\pi_N"] & & \\ M \arrow[r, "\alpha"] \arrow[dr, "j_M"'] & N \arrow[ur, "j_N"] \arrow[rr, "\beta"] & & M \arrow[r, "\alpha"] & N \\ & P_0(M) \arrow[urr, "\pi_M"'] & & & \end{tikzcd}$$

where $1_M - \beta\alpha = \pi_M j_M$ and $1_N - \alpha\beta = \pi_N j_N$. Furthermore, since π_M and π_N are epimorphisms, there exist morphisms $f_N : P_0(N) \to P_0(M)$ and $f_M : P_0(M) \to P_0(N)$ in $\mathrm{mod}\,(A)$ satisfying the equalities $\pi_M f_N = \beta\pi_N$ and $\pi_N f_M = \alpha\pi_M$. So we get the following diagram in $\mathrm{mod}\,(A)$

$$M \oplus P_0(N) \xrightarrow{F} N \oplus P_0(M) \xrightarrow{G} M \oplus P_0(N),$$

where $F := \begin{pmatrix} \alpha & \pi_N \\ j_M & -f_N \end{pmatrix}$ and $G := \begin{pmatrix} \beta & \pi_M \\ j_N & -f_M \end{pmatrix}$. We assert that FG and GF are isomorphisms. Indeed, let $\mu := j_M\pi_M + f_N f_M \in \mathrm{End}_A(P_0(M))$ and so, from the above equalities, we have that $FG = \begin{pmatrix} 1_N & 0 \\ j_M\beta - f_N j_N & \mu \end{pmatrix}$. Hence, in order to prove that FG is an isomorphism, it is enough to check that $\pi_M\mu = \pi_M$ since π_M is a right-minimal morphism. That is, $\pi_M\mu = \pi_M j_M\pi_M + \pi_M f_N f_M = \pi_M j_M \pi_M + \beta\pi_N f_M = \pi_M j_M\pi_M + \beta\alpha\pi_M = (\pi_M j_M + \beta\alpha)\pi_M = 1_M\pi_M = \pi_M$; proving that FG is an isomorphism. Analogously, it can be seen that GF is an isomorphism.

(b) $\Rightarrow$ (c) It is straightforward.

(c) $\Leftrightarrow$ (d) It follows from the fact that $\mathrm{proj}\,(A)$ is the iso-class of the zero object in the stable category $\underline{\mathrm{mod}}\,(A)$. $\square$

For an artin R-algebra A, we denote by $\mathcal{E}(A)$ the quotient of the free abelian group generated by the iso-classes $[\mathrm{Ext}^1_A(M,-)]$ in $\mathcal{C}_A$, for all $M \in \mathrm{mod}\,(A)$, modulo the relations

$$[\mathrm{Ext}^1_A(N,-)] - [\mathrm{Ext}^1_A(X,-)] - [\mathrm{Ext}^1_A(Y,-)] \quad \text{if} \quad N \simeq X \oplus Y.$$

The syzygy functor $\Omega : \underline{\mathrm{mod}}\,(A) \to \underline{\mathrm{mod}}\,(A)$ gives rise to a group homomorphism $\Omega : \mathcal{E}(A) \to \mathcal{E}(A)$, given by $\Omega([\mathrm{Ext}^1_A(M,-)]) := [\mathrm{Ext}^1_A(\Omega\, M,-)]$.

Theorem 3.2. *Let A be an artin R-algebra. Then, the following statements hold.*

(a) *The map $\varepsilon : \mathrm{K}(A) \to \mathcal{E}(A)$, given by $\varepsilon([M]) := [\mathrm{Ext}^1_A(M,-)]$, is an isomorphism of abelian groups and the following diagram is commutative*

$$\begin{tikzcd} \mathrm{K}(A) \arrow[r, "\varepsilon"] \arrow[d, "\Omega"'] & \mathcal{E}(A) \arrow[d, "\Omega"] \\ \mathrm{K}(A) \arrow[r, "\varepsilon"'] & \mathcal{E}(A). \end{tikzcd}$$

(b) $\varepsilon(\Omega^n([M])) = [\mathrm{Ext}^{n+1}_A(M,-)]$ *for any $n \in \mathbb{N}$ and $M \in \mathrm{mod}\,(A)$.*

Proof. (a) Since the bifunctor $\mathrm{Ext}^1_A(-,-)$ commutes with finite direct sums and takes, in the first variable, objects of $\mathrm{proj}\,(A)$ to zero, we get that the surjective

map $\varepsilon : \mathrm{K}(A) \to \mathcal{E}(A)$ is well defined and it is also a morphism of abelian groups. Now, let $M, N \in \mathrm{mod}\,(A)$ be such that $\varepsilon\,([M]) = \varepsilon\,([N])$. Then, by Proposition 3.1, we get that $[M] = [N]$ in $\mathrm{K}(A)$; proving that ε is an isomorphism. Finally, we have $\varepsilon(\Omega\,([M]) = [\mathrm{Ext}^1_A(\Omega\, M, -)] = \Omega(\varepsilon([M]))$.

(b) $\varepsilon(\Omega^n([M]) = [\mathrm{Ext}^1_A(\Omega^n\, M, -)] = [\mathrm{Ext}^{n+1}_A(M, -)]$. □

Corollary 3.3. *Let A be an artin R-algebra and $M, N \in \mathrm{mod}(A)$. Then, the following conditions are equivalent.*

(a) $[\Omega^n M] = [\Omega^n N]$ *in* $\mathrm{K}(A)$.
(b) $\mathrm{Ext}^i_A(M, -) \simeq \mathrm{Ext}^i_A(N, -)$ *in* $\mathcal{C}_A$ *for any* $i \geq n + 1$.
(c) $\mathrm{Ext}^{n+1}_A(M, -) \simeq \mathrm{Ext}^{n+1}_A(N, -)$ *in* $\mathcal{C}_A$.

Proof. It follows easily from Theorem 3.2. □

Definition 3.4. *Let A be an artin R-algebra, d be a positive integer and M in* $\mathrm{mod}\,(A)$. *A pair (X, Y) of objects in* $\mathrm{add}\,(M)$ *is called a d-Division of M if the following three conditions hold:*

(a) $\mathrm{add}\,(X) \cap \mathrm{add}\,(Y) = \{0\}$;
(b) $\mathrm{Ext}^d_A(X, -) \not\simeq \mathrm{Ext}^d_A(Y, -)$ *in* $\mathcal{C}_A$;
(c) $\mathrm{Ext}^{d+1}_A(X, -) \simeq \mathrm{Ext}^{d+1}_A(Y, -)$ *in* $\mathcal{C}_A$.

Remark 3.5. *Observe that $\phi\,(M) = 0$ if and only if for any pair (X, Y) of objects in* $\mathrm{add}\,(M)$, *which are not projective and* $\mathrm{add}\,(X) \cap \mathrm{add}\,(Y) = \{0\}$, *we have that* $\mathrm{Ext}^d_A(X, -) \not\simeq \mathrm{Ext}^d_A(Y, -)$ *in $\mathcal{C}_A$ for any $d \geq 1$. Thus, in this case, the following set is empty*

$$\{d \in \mathbb{N} \;:\; \textit{there is a d-Division of } M\}.$$

The following result gives a characterization of $\phi\,(M)$ in terms of the bifunctors $\mathrm{Ext}^i_A(-, -)$.

Theorem 3.6. *Let A be an artin R-algebra and M in* $\mathrm{mod}\,(A)$. *Then*

$$\phi\,(M) = \max\,(\{d \in \mathbb{N} \;:\; \textit{there is a d-Division of } M\} \cup \{0\}).$$

Proof. Let $n := \phi\,(M) > 0$ and $M = \oplus_{i=1}^t M_i^{m_i}$ be a direct sum decomposition of M, where $M_i \not\simeq M_j$ for $i \neq j$ and M_i is indecomposable for any i. Since n is the first moment from which the rank of each free abelian group of the family $\{\Omega^j(\langle M\rangle) \;:\; j \in \mathbb{N}\}$ becomes stable, e.g.

$$\phi\,(M) = \min\,\{m \in \mathbb{N} \;:\; \mathrm{rk}\,\Omega^j(\langle M\rangle) = \mathrm{rk}\,\Omega^m(\langle M\rangle)\;\forall\, j \geq m\},$$

we get the existence of natural numbers $\alpha_1, \cdots, \alpha_t$ (not all zero) and a partition $\{1, 2, \cdots, t\} = I \uplus J$ such that $\sum_{i\in I} \alpha_i[\Omega^n M_i] = \sum_{j\in J} \alpha_j[\Omega^n M_j]$, and furthermore $\sum_{i\in I} \alpha_i[\Omega^{n-1} M_i] \neq \sum_{j\in J} \alpha_j[\Omega^{n-1} M_j]$.

Hence, by Corollary 3.3, it follows that the pair (X, Y), with $X := \oplus_{i\in I} M_i^{\alpha_i}$ and $Y := \oplus_{j\in J} M_j^{\alpha_j}$, is an n-Division of M. Moreover, by using the fact that $\mathrm{rk}\,\Omega^j(\langle M\rangle) > \mathrm{rk}\,\Omega^n(\langle M\rangle)$ for $j = 0, 1, \cdots, n-1$, we get the result. □

4. Invariance of the ϕ-dimension

Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathfrak{A}$. A complex $X^\bullet$ over $\mathcal{A}$ is a sequence of morphisms $\{d_X^i : X^i \to X^{i+1}\}_{i\in\mathbb{Z}}$ in $\mathcal{A}$, called the differentials of the complex $X^\bullet$, such that $d_X^i d_X^{i-1} = 0$ for all $i \in \mathbb{Z}$. We write $X^\bullet = (X^i, d_X^i)$ and denote by $H^i(X^\bullet)$ the i-th cohomology of the complex $X^\bullet$. Observe that $H^\bullet(X^\bullet)$, with zero differentials, is also a complex over $\mathfrak{A}$. Usually, a complex $X^\bullet$ is written as follows

$$X^\bullet \;:\quad \cdots \longrightarrow X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \longrightarrow \cdots$$

A complex $X^\bullet$ induces, in a natural way, the following complexes (called truncations)

$$\tau_{\leq n}(X^\bullet) \;:\quad \cdots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \cdots ,$$

$$\tau_{\geq n}(X^\bullet) \;:\quad \cdots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \cdots .$$

The category of all complexes over $\mathcal{A}$, with the usual complex maps of degree zero as morphisms, is denoted by $\mathrm{C}(\mathcal{A})$. Hence, we have the i-th cohomology functor $H^i : \mathrm{C}(\mathcal{A}) \to \mathfrak{A}$. For a given subset Ξ of $\mathbb{Z}$, we consider the class of complexes

$$\mathrm{C}(\mathcal{A})_\Xi := \{\; X^\bullet \in \mathrm{C}(\mathcal{A}) \;:\quad X^i = 0 \quad \forall\, i \notin \Xi\}$$

which are concentrated on the set Ξ. Usually, as Ξ, we shall consider intervals of integer numbers of the form $[n,m] := \{x \in \mathbb{Z} \;:\; n \leq x \leq m\}$.

We denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of complexes over $\mathcal{A}$, and by $\mathrm{K}^-(\mathcal{A})$, $\mathrm{K}^+(\mathcal{A})$ and $\mathrm{K}^b(\mathcal{A})$ to the full triangulated subcategories of $\mathrm{K}(\mathcal{A})$ consisting, respectively, of the bounded above, bounded below and bounded complexes. In case $\mathcal{A}$ is an abelian category, we denote by $\mathrm{D}(\mathcal{A})$ the derived category of complexes over $\mathcal{A}$, and by $\mathrm{D}^-(\mathcal{A})$, $\mathrm{D}^+(\mathcal{A})$ and $\mathrm{D}^b(\mathcal{A})$ the full triangulated subcategories of $\mathrm{D}(\mathcal{A})$ consisting, respectively, of the complexes which are bounded above, bounded below and with bounded cohomology.

Let Λ be an artin R-algebra. In this special case, we simplify the notation by writting $\mathrm{D}(\Lambda)$ instead of $\mathrm{D}(\mathrm{mod}(\Lambda))$. Analogously, we write $\mathrm{C}(\Lambda)$ instead of $\mathrm{C}(\mathrm{mod}(\Lambda))$. Furthermore, for any integers a, b with $a \leq b$, we denote by $\mathrm{D}(\Lambda)_{[a,b]}$ to the full subcategory of $\mathrm{D}(\Lambda)$ whose objects $X^\bullet$ are such that $X^i = 0$ for $i \notin [a,b]$. It is also well-known that the canonical functor $\imath_0 : \mathrm{mod}(\Lambda) \to \mathrm{D}(\Lambda)$, which sends $M \in \mathrm{mod}(\Lambda)$ to the stalk complex $M[0]$ concentrated in degree zero, is additive full and faithful. Hence, through the functor $\imath_0$, the module category $\mathrm{mod}(\Lambda)$ can be considered as a full additive subcategory of $\mathrm{D}(\Lambda)$. Finally, for the sake of simplicity, we sometimes denote the bifunctor $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(-,-)$ by $(-,-)$.

In order to prove the main result of this section, we start with the following preparatory lemmas. In all that follows, Λ stands for an artin R-algebra.

Lemma 4.1. [10, Lemma 1.6] *Let $m, k, d \in \mathbb{Z}$ with $d \geq 0$; and let $X^\bullet, Y^\bullet \in \mathrm{K}^b(\mathrm{mod}(\Lambda))$ be such that $X^p = 0$ for all $p < m$ and $Y^q = 0$ for all $q > k$. If $\mathrm{Ext}^i_\Lambda(X^r, Y^s) = 0$ for all $r, s \in \mathbb{Z}$ and $i \geq d$, then*

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(X^\bullet, Y^\bullet[i]) = 0 \quad \textit{for all } i \geq d + k - m.$$

Lemma 4.2. *Let $k \geq 0$ and $\ell > 0$; and let $Z^\bullet, W^\bullet \in \mathrm{C}(\Lambda)_{[-k,0]}$ with $Z^i, W^i \in \mathrm{proj}(\Lambda)$ $\forall\, i \in [-k+1, 0]$.*
If $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(Z^\bullet, -[t])|_{\mathrm{mod}(\Lambda)} \simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(W^\bullet, -[t])|_{\mathrm{mod}(\Lambda)}$ for all $t \geq k + \ell$, then

$$\mathrm{Ext}^t_\Lambda(Z^{-k}, -) \simeq \mathrm{Ext}^t_\Lambda(W^{-k}, -) \;\; \textit{for all} \;\; t \geq k + \ell.$$

Proof. Assume that $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(Z^\bullet, -[t])|_{\mathrm{mod}(\Lambda)} \simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(W^\bullet, -[t])|_{\mathrm{mod}(\Lambda)}$ for all $t \geq k+\ell$. Let $S \in \mathrm{mod}(\Lambda)$. By applying the functor $(-, S[t])$ to the distinguished triangles

$$Z^0[0] \to Z^\bullet \to \tau_{\leq -1}(Z^\bullet) \to Z^0[1] \text{ and } W^0[0] \to W^\bullet \to \tau_{\leq -1}(W^\bullet) \to W^0[1],$$

we get the following exact sequences

$$(Z^0[0], S[t]) \to (\tau_{\leq -1}(Z^\bullet)[-1], S[t]) \to (Z^\bullet[-1], S[t]) \to (Z^0[-1], S[t]),$$

$$(W^0[0], S[t]) \to (\tau_{\leq -1}(W^\bullet)[-1], S[t]) \to (W^\bullet[-1], S[t]) \to (W^0[-1], S[t]).$$

Moreover, since Z^0, $W^0 \in \mathrm{proj}(\Lambda)$ and $t \geq k+\ell \geq 1$, it follows that $(\tau_{\leq -1}(Z^\bullet)[-1], -[t])|_{\mathrm{mod}(\Lambda)} \simeq (Z^\bullet[-1], -[t])|_{\mathrm{mod}(\Lambda)}$ and also we have that $(\tau_{\leq -1}(W^\bullet)[-1], -[t])|_{\mathrm{mod}(\Lambda)} \simeq (W^\bullet[-1], -[t])|_{\mathrm{mod}(\Lambda)}$ for any $t \geq n+\ell$. Hence, by composing the given above isomorphisms of functors, we get that

$$(\tau_{\leq -1}(Z^\bullet)[-1], -[t])|_{\mathrm{mod}(\Lambda)} \simeq (\tau_{\leq -1}(W^\bullet)[-1], -[t])|_{\mathrm{mod}(\Lambda)} \text{ for any } t \geq k+\ell.$$

We proceed by induction as follows. The step $i = 1$, is given by the above isomorphism of functors. Consider $1 < i \leq k-1$ and apply the functor $(-, S[t])$, for each $S \in \mathrm{mod}(\Lambda)$, to the distinguished triangles $Z^{-i}[0] \to \tau_{\leq -i}(Z^\bullet)[-i] \to \tau_{\leq -i-1}(Z^\bullet)[-i] \to Z^{-i}[1]$ and $W^{-i}[0] \to \tau_{\leq -i}(W^\bullet)[-i] \to \tau_{\leq -i-1}(W^\bullet)[-i] \to W^{-i}[1]$. So, by repeating the same argument as above and using that $(\tau_{\leq -i}(Z^\bullet)[-i], -[t])|_{\mathrm{mod}(\Lambda)} \simeq (\tau_{\leq -i}(Z^\bullet)[-i], -[t])|_{\mathrm{mod}(\Lambda)}$ (the previous step) for all $t \geq k+d$, the fact that Z^{-i} and W^{-i} are projective for all $-i \in [-k+1, 0]$, we obtain that $(Z^{-k}[0], -[t])|_{\mathrm{mod}(\Lambda)} \simeq (W^{-k}[0], -[t])|_{\mathrm{mod}(\Lambda)}$ for all $t \geq k+d$. □

Lemma 4.3. *Let k, $\ell \in \mathbb{Z}$ be such that $0 < k < \ell$. Let $Z^\bullet, W^\bullet \in \mathrm{D}(\Lambda)_{[-k,0]}$ with Z^i, $W^i \in \mathrm{proj}(\Lambda)$ for all $i \in [-k+1, 0]$.*
If $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(Z^{-k}[k], -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}} \simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(W^{-k}[k], -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}}$, then

$$\mathrm{Hom}_{\mathrm{D}(\Lambda)}(Z^\bullet, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}} \simeq \mathrm{Hom}_{\mathrm{D}(\Lambda)}(W^\bullet, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}}.$$

Proof. For simplicity, we write $(-,-)$ instead of $\mathrm{Hom}_{\mathrm{D}(\Lambda)}(-,-)|_{\mathrm{D}(\Lambda)_{[-k,0]}}$. Suppose that $(Z^\bullet, -[\ell]) \not\simeq (W^\bullet, -[\ell])$. From the following distinguished triangles

$$Z^0[0] \to Z^\bullet \to \tau_{\leq -1}(Z^\bullet) \to Z^0[1] \text{ and } W^0[0] \to W^\bullet \to \tau_{\leq -1}(W^\bullet) \to W^0[1],$$

we get the following exact sequences of functors

$$(Z^0[1], -[\ell]) \to (\tau_{\leq -1}(Z^\bullet), -[\ell]) \to (Z^\bullet, -[\ell]) \to (Z^0[0], -[\ell]),$$

$$(W^0[1], -[\ell]) \to (\tau_{\leq -1}(W^\bullet), -[\ell]) \to (W^\bullet, -[\ell]) \to (W^0[0], -[\ell]).$$

From the Lemma 4.1, we have that $(Z^0[1], -[\ell]) = (Z^0[0], -[\ell]) = (W^0[1], -[\ell]) = (W^0[0], -[\ell]) = 0$. Therefore $(\tau_{\leq -1}(Z^\bullet), -[\ell]) \simeq (Z^\bullet, -[\ell])$ and $(\tau_{\leq -1}(W^\bullet), -[\ell]) \simeq (W^\bullet, -[\ell])$. By our assumption, we get that

$$(\tau_{\leq -1}(Z^\bullet), -[\ell]) \not\simeq (\tau_{\leq -1}(W^\bullet), -[\ell]).$$

We proceed by induction as follows. The step $i = 1$ is given by the above statement. So, by the inductive step, we can assume that $(\tau_{\leq -i}(Z^\bullet), -[\ell]) \not\simeq (\tau_{\leq -i}(W^\bullet), -[\ell])$ for any $1 < i \leq k-1$. By applying the functor $(\,, -[\ell])$ to the distinguished triangles $Z^{-i}[i] \to \tau_{\leq -i}(Z^\bullet) \to \tau_{\leq -i-1}(Z^\bullet) \to Z^{-i}[i+1]$ and $W^{-i}[i] \to \tau_{\leq -i}(W^\bullet) \to \tau_{\leq -i}(W^\bullet) \to W^{-i}[i+1]$, and using that $(Z^{-i}[i+1], -[\ell]) = (W^{-i}[i+1], -[\ell]) = (Z^{-i}[i], -[\ell]) = (W^{-i}[i], -[\ell]) = 0$, we get that

$$(\tau_{\leq -i-1}(Z^\bullet), -[\ell]) \not\simeq (\tau_{\leq -i-1}(W^\bullet), -[\ell]).$$

Thus, we obtain $(Z^{-k}[k], -[\ell]) \not\simeq (W^{-k}[k], -[\ell])$; which is a contradiction. So, our assumption is not true, and hence the result follows. $\square$

Lemma 4.4. *Let $\ell > 0$ and $k \geq 0$; and let $K,\ S \in \operatorname{mod}(\Lambda)$. If $\operatorname{Ext}^{\ell}_{\Lambda}(K, -) \simeq \operatorname{Ext}^{\ell}_{\Lambda}(S, -)$ then $\operatorname{Hom}_{\mathrm{D}(\Lambda)}(K, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}} \simeq \operatorname{Hom}_{\mathrm{D}(\Lambda)}(S, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}}$.*

Proof. By [3] (see Section 1 and Section 2), we have that there is an isomorphism $\mu_{Y^\bullet} : Y^\bullet \to L(Y^\bullet)$ in $\mathrm{D}(\Lambda)_{[-k,0]}$, which is natural on the variable $Y^\bullet$; and moreover $L(Y^\bullet)^i \in \operatorname{inj}(\Lambda)\ \ \forall\, i \in [-k, -1]$.

Let $Y^\bullet \in \mathrm{D}(\Lambda)_{[-k,0]}$. Then, by the discussion, given above, it can be assumed that $Y^i \in \operatorname{inj}(\Lambda)\ \ \forall\, i \in [-k, -1]$. By applying the functors

$$(K, -) := \operatorname{Hom}_{\mathrm{D}(\Lambda)}(K, -)|_{\mathrm{D}(\Lambda)_{[-k,0]}} \text{ and } (S, -) := \operatorname{Hom}_{\mathrm{D}(\Lambda)}(S, -)|_{\mathrm{D}(\Lambda)_{[-k,0]}}$$

to the distinguished triangle $Y^0[\ell] \to Y^\bullet[\ell] \to \tau_{\leq -1}(Y^\bullet)[\ell] \to Y^0[\ell]$, we obtain the following exact sequences

$$(K, \tau_{\leq -1}(Y^\bullet)[\ell-1]) \to (K, Y^0[\ell]) \to (K, Y^\bullet[\ell]) \to (K, \tau_{\leq -1}(Y^\bullet)[\ell]),$$

$$(S, \tau_{\leq -1}(Y^\bullet)[\ell-1]) \to (S, Y^0[\ell]) \to (S, Y^\bullet[\ell]) \to (S, \tau_{\leq -1}(Y^\bullet)[\ell]).$$

By Lemma 4.1, we have that the following equalities hold $(K, \tau_{\leq -1}(Y^\bullet)[\ell-1]) = (S, \tau_{\leq -1}(Y^\bullet)[\ell-1]) = (K, \tau_{\leq -1}(Y^\bullet)[\ell]) = (S, \tau_{\leq -1}(Y^\bullet)[\ell]) = 0$; and since $\operatorname{Ext}^{\ell}_{\Lambda}(K, -) \simeq \operatorname{Ext}^{\ell}_{\Lambda}(S, -)$, it follows that $(K, -[\ell]) \simeq (S, -[\ell])$, proving the result. $\square$

Corollary 4.5. *Let k and ℓ be integers such that $0 < k < \ell$; and let $Z^\bullet, W^\bullet \in \mathrm{D}(\Lambda)_{[-k,0]}$ be such that $Z^i, W^i \in \operatorname{proj}(\Lambda)\ \ \forall\, i \in [-k+1, 0]$.*
If $\operatorname{Ext}^{\ell-k}_{\Lambda}(Z^{-k}, -) \simeq \operatorname{Ext}^{\ell-k}_{\Lambda}(W^{-n}, -)$ then

$$\operatorname{Hom}_{\mathrm{D}(\Lambda)}(Z^\bullet, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}} \simeq \operatorname{Hom}_{\mathrm{D}(\Lambda)}(W^\bullet, -[\ell])|_{\mathrm{D}(\Lambda)_{[-k,0]}}.$$

Proof. The proof follows from Lemma 4.3 and Lemma 4.4. $\square$

We recall now some definitions (see [13]). A tilting complex over an artin algebra A is a complex $T^\bullet \in \mathrm{K}^b(\operatorname{proj}(A))$ such that $\operatorname{Hom}_{\mathrm{K}^b(\operatorname{proj}(A))}(T, T[n]) = 0$ for all integers $n \neq 0$, and $\operatorname{add}(T^\bullet)$, the full subcategory of direct summands of finite direct sums of copies of $T^\bullet$, generates $\mathrm{K}^b(\operatorname{proj}(A))$ as a triangulated category. Two artin algebras A and B are called derived equivalent if $\mathrm{D}(A)$ and $\mathrm{D}(B)$ are equivalent as triangulated categories. For further properties of derived equivalent artin algebras, the reader can see in [13].

In all that follows, we shall consider the following situation: Let A be an artin algebra, $T^\bullet \in \mathrm{K}^b(\operatorname{proj}(A))$ a tilting complex, $B := \operatorname{End}_{\mathrm{D}(A)}(T^\bullet)^{op}$ and $F : \mathrm{D}(B) \to \mathrm{D}(A)$ be an equivalence of triangulated categories such that $F(B) = T^\bullet$. The quasi-inverse of F is denoted by $G : \mathrm{D}(A) \to \mathrm{D}(B)$. It is well known that F induces equivalences $\mathrm{D}^b(B) \to \mathrm{D}^b(A)$ and $\mathrm{K}^b(\operatorname{proj}(B)) \to \mathrm{K}^b(\operatorname{proj}(A))$; and furthermore, without loss of generality, it can be assumed that $T^\bullet \in \mathrm{C}(\operatorname{proj}(A))_{[-n,0]}$ for some $n \geq 0$.

The following results are well known in the literature, for a proof the reader can see in [10, 12].

Lemma 4.6. [10, 12] *Let $B := \operatorname{End}_{\mathrm{D}(A)}(T^\bullet)^{op}$, where $T^\bullet \in \mathrm{C}(\operatorname{proj}(A))_{[-n,0]}$ is a tilting complex as above. Then, the following statements hold.*

(a) *There is a tilting complex* $Q^\bullet \in \mathrm{C}\,(\mathrm{proj}\,(B))_{[0,n]}$ *such that* $G(A) \simeq Q^\bullet$.
(b) $H^\bullet(FM) \in \mathrm{C}\,(A)_{[-n,0]}$ *for any* $M \in \mathrm{mod}\,(A)$.
(c) $H^\bullet(GN) \in \mathrm{C}\,(B)_{[0,n]}$ *for any* $N \in \mathrm{mod}\,(B)$.

Lemma 4.7. *Let* $B := \mathrm{End}_{\mathrm{D}\,(A)}(T^\bullet)^{op}$, *where* $T^\bullet \in \mathrm{C}\,(\mathrm{proj}\,(A))_{[-n,0]}$ *is a tilting complex as above. Then, for any* $M \in \mathrm{mod}\,(B)$, *the complex* FM *is quasi-isomorphic to complexes* $X^\bullet_M$ *and* $Y^\bullet_M$, *where*

(a) $X^\bullet_M \in \mathrm{C}\,(A)_{[-n,0]}$ *with* $X^i_M \in \mathrm{proj}\,(A)\ \ \forall\, i \in [-n+1, 0]$; *and*
(b) $Y^\bullet_M \in \mathrm{C}\,(A)_{[-n,0]}$ *with* $Y^i_M \in \mathrm{inj}\,(A)\ \ \forall\, i \in [-n, -1]$.

Moreover, those quasi-isomorphisms, given above, induce natural isomorphisms in the derived category $\mathrm{D}\,(A)$.

Proof. (a) Consider the complex $P^\bullet(M)$ induced by the minimal projective resolution of M. Since F induces an equivalence $\mathrm{K}^-\,(\mathrm{proj}\,(B)) \to \mathrm{K}^-\,(\mathrm{proj}\,(A))$, it follows that $FP^\bullet(M) \in \mathrm{K}^-\,(\mathrm{proj}\,(A))$. By Lemma 4.6 (b), we thave that $H^\bullet(FP^\bullet(M)) \simeq H^\bullet(FM) \in \mathrm{C}\,(A)_{[-n,0]}$. Thus, the complex $Z^\bullet := FP^\bullet(M)$ is quasi-isomorphic to the following complex $C^\bullet \in \mathrm{C}\,(A)_{[-n,0]}$, where

$$C^\bullet\ :\ \cdots \to \mathrm{Coker}\,(d^{-n-1}) \to Z^{-n+1} \to \cdots \to Z^{-1} \to \mathrm{Ker}\,(d^0) \to \cdots .$$

We assert that $\mathrm{Ker}\,(d^0) \in \mathrm{proj}\,(A)$. Indeed, since $H^i(Z^\bullet) = 0$ for $i > 0$, the complex

$$\cdots \to 0 \to \mathrm{Ker}\,(d^0) \to Z^0 \to Z^1 \to Z^2 \to \cdots \tag{1}$$

is exact. Moreover, the complex $Z^\bullet$ is bounded above and each Z^i is projective, and so the complex (1) splits; proving that $\mathrm{Ker}\,(d^0) \in \mathrm{proj}\,(A)$. Therefore, the complex $X^\bullet_M := C^\bullet$ satisfies the desired conditions.

(b) This can be proven in a similar way as we did in (a). □

Lemma 4.8. *Let* $B := \mathrm{End}_{\mathrm{D}\,(A)}(T^\bullet)^{op}$, *where* $T^\bullet \in \mathrm{C}\,(\mathrm{proj}\,(A))_{[-n,0]}$ *is a tilting complex as above. Then, for any* $S \in \mathrm{mod}\,(A)$, *the complex* GS *is quasi-isomorphic to complexes* $X^\bullet_S$ *and* $Y^\bullet_S$, *where*

(a) $X^\bullet_S \in \mathrm{C}\,(B)_{[0,n]}$ *with* $X^i_S \in \mathrm{proj}\,(B)\ \ \forall\, i \in [1, n]$; *and*
(b) $Y^\bullet_S \in \mathrm{C}\,(B)_{[0,n]}$ *with* $Y^i_S \in \mathrm{inj}\,(B)\ \ \forall\, i \in [0, n-1]$.

Moreover, those quasi-isomorphisms induce natural isomorphisms in the derived category $\mathrm{D}\,(B)$.

Proof. It can be done, in an similar way, as we did in 4.7. □

Lemma 4.9. *Let* $B := \mathrm{End}_{\mathrm{D}\,(A)}(T^\bullet)^{op}$, *where* $T^\bullet \in \mathrm{C}\,(\mathrm{proj}\,(A))_{[-n,0]}$ *is a tilting complex as above; and let* $M, N \in \mathrm{mod}\,(B)$. *If* $\mathrm{Ext}^t_B(M, -) \simeq \mathrm{Ext}^t_B(N, -)$ *for all* $t \geq \ell > 0$, *then*

$$\mathrm{Hom}_{\mathrm{D}\,(B)}(M,\, G(-[t]))|_{\mathrm{mod}\,(A)} \simeq \mathrm{Hom}_{\mathrm{D}\,(B)}(N,\, G(-[t]))|_{\mathrm{mod}\,(A)} \quad \forall\, t \geq n + \ell.$$

Proof. Let $S \in \mathrm{mod}\,(A)$. Thus, by Lemma 4.8 (b), we have the natural isomorphism $GS \xrightarrow{\sim} Y^\bullet_S$ in $\mathrm{D}\,(B)$, where $Y^\bullet_S \in \mathrm{C}\,(B)_{[0,n]}$ and $Y^i_S \in \mathrm{inj}\,(B)\ \ \forall\, i \in [0, n-1]$. Applying the functors $(M, -)$ and $(N, -)$ to the distinguished triangle

$$Y^n_S[-n+t] \to Y^\bullet_S[t] \to \tau_{\leq n-1}(Y^\bullet_S)[t] \to Y^n_S[-n+t+1],$$

we obtain, for any $t \geq n + \ell$, the following exact sequences

$$(M, \tau_{\leq n-1}(Y^\bullet_S)[t-1]) \to (M, Y^n_S[t-n]) \to (M, Y^\bullet_S[t]) \to (M, \tau_{\leq n-1}(Y^\bullet_S)[t]),$$

$$(N, \tau_{\leq n-1}(Y^\bullet_S)[t-1]) \to (N, Y^n_S[t-n]) \to (N, Y^\bullet_S[t]) \to (N, \tau_{\leq n-1}(Y^\bullet_S)[t]).$$

By Lemma 4.1, we have that the following equalities hold $(M, \tau_{\leq n-1}(Y_S^\bullet)[t-1]) = (M, \tau_{\leq n-1}(Y_S^\bullet)[t]) = (N, \tau_{\leq n-1}(Y_S^\bullet)[t-1]) = (N, \tau_{\leq n-1}(Y_S^\bullet)[t]) = 0$, and thus we get that $(M, Y_S^n[t-n]) \simeq (M, Y_S^\bullet[t])$ and $(N, Y_S^n[t-n]) \simeq (N, Y_S^\bullet[t])$ for $t \geq n+\ell$. Therefore $(M, G(-[t]))|_{\mathrm{mod}\,(A)} \simeq (N, G(-[t]))|_{\mathrm{mod}\,(A)}$ for all $t \geq n+\ell$. □

For an artin algebra Λ, we denote the ϕ dimension of a Λ-module M by $\phi_\Lambda(M)$.

Theorem 4.10. *Let A and B be artin algebras, which are derived equivalent. Then, $\phi\dim(A) < \infty$ if and only if $\phi\dim(B) < \infty$. More precisely, if $T^\bullet$ is a tilting complex over A with n non-zero terms and such that $B \simeq \mathrm{End}_{\mathrm{D}\,(A)}(T^\bullet)$, then*

$$\phi\dim(A) - n \leq \phi\dim(B) \leq \phi\dim(A) + n.$$

Proof. Assume that $\phi\dim(A) = r < \infty$. If $\phi\dim(B) \leq n$ then $\phi\dim(B) \leq \phi\dim(A)+n$. Suppose now that there exists $J \in \mathrm{mod}\,(B)$ such that $\phi_B(J) = d > n$. It follows from Theorem 3.6 that $J = \hat{M} \oplus \hat{N}$ and there exist $M \in \mathrm{add}\,(\hat{M})$ and $N \in \mathrm{add}\,(\hat{N})$ such that

$$\begin{cases} \mathrm{Ext}_B^d(M,-) & \not\simeq & \mathrm{Ext}_B^d(N,-), \\ \mathrm{Ext}_B^t(M,-) & \simeq & \mathrm{Ext}_B^t(N,-) \quad \text{for } t \geq d+1. \end{cases} \tag{2}$$

Using Lemma 4.9, for the second equation of (2), we have

$$\begin{cases} \mathrm{Hom}_{\mathrm{D}\,(B)}(M,-[d])|_{\mathrm{mod}\,(B)} & \not\simeq & \mathrm{Hom}_{\mathrm{D}\,(B)}(N,-[d])|_{\mathrm{mod}\,(B)}, \\ \mathrm{Hom}_{\mathrm{D}\,(B)}(M,G(-[t]))|_{\mathrm{mod}\,(A)} & \simeq & \mathrm{Hom}_{\mathrm{D}\,(B)}(N,G(-[t]))|_{\mathrm{mod}\,(A)} \end{cases} \tag{3}$$

for all $t \geq n+d+1$.
Applying the equivalence F in (3), we get

$$\begin{cases} \mathrm{Hom}_{\mathrm{D}\,(A)}(FM,F(-[d]))|_{\mathrm{mod}\,(A)} & \not\simeq & \mathrm{Hom}_{\mathrm{D}\,(A)}(FN,F(-[d]))|_{\mathrm{mod}\,(A)}, \\ \mathrm{Hom}_{\mathrm{D}\,(A)}(FM,-[t])|_{\mathrm{mod}\,(A)} & \simeq & \mathrm{Hom}_{\mathrm{D}\,(A)}(FN,-[t])|_{\mathrm{mod}\,(A)} \end{cases} \tag{4}$$

for all $t \geq n+d+1$.

By Lemma 4.7, the complexes FM and FN can be replaced, respectively, by $Z^\bullet, W^\bullet \in \mathrm{C}\,(A)_{[-n,0]}$ such that $Z^i, W^i \in \mathrm{proj}\,(A)$ $\forall\, i \in [-n+1, 0]$. It follows, from Lemma 4.2, that the second item of (4) is equivalent to

$$\mathrm{Ext}_A^t(Z^{-n},-) \simeq \mathrm{Ext}_A^t(W^{-n},-) \tag{5}$$

for all $t \geq n+d+1$.

By Corollary 4.5, the first item of (4) give us that

$$\mathrm{Ext}_A^{d-n}(Z^{-n},-) \not\simeq \mathrm{Ext}_A^{d-n}(W^{-n},-). \tag{6}$$

In particular, we get from (6) that $Z^{-n} \not\simeq W^{-n}$. We can assume that $\mathrm{add}\,(Z^{-n})$ and $\mathrm{add}\,(W^{-n})$ have trivial intersection because, otherwise, we can decompose Z^{-n} and W^{-n} as direct sum of indecomposables and withdraw from each one the common factors. The modules obtained satisfy (5) and (6), and their additive closure has trivial intersection.

Let $L := Z^{-n} \oplus W^{-n}$. It follows from Theorem 3.6, (5) and (6) that $d - n \leq \phi_A(L) \leq d+n$. Since $\phi\dim(A) = r$. we have $d \leq n+r$ and hence $\phi\dim(B) \leq n + \phi\dim(A)$. Analogously, it can be shown that $\phi\dim(A) \leq \phi\dim(B) + n$. □

As an application of the main result, we get the following generalisation of the classic Bongartz's result [2, Corollary 1]. In order to do that, we recall firstly, the notion of tilting module. Following Y. Miyashita in [11], it is said that an A-module $T \in \mathrm{mod}\,(A)$ is a tilting module, if T satisfies the following properties: (a) $\mathrm{pd}\, T$

is finite, (b) $\mathrm{Ext}^i_A(T,T) = 0$ for any $i > 0$ and (c) there is an exact sequence $0 \to {}_AA \to T_0 \to T_1 \to \cdots \to T_m \to 0$ in $\mathrm{mod}\,(A)$, with $T_i \in \mathrm{add}\,(T)$ for any $0 \leq i \leq m$.

Corollary 4.11. *Let A be an artin algebra, and let $T \in \mathrm{mod}\,(A)$ be a tilting A-module. Then, for the artin algebra $B := \mathrm{End}_A(T)^{op}$, we have that*

$$\phi\dim(A) - \mathrm{pd}\,T \leq \phi\dim(B) \leq \phi\dim(A) + \mathrm{pd}\,T.$$

Proof. Let $m := \mathrm{pd}\,T$. Then, the minimal projective resolution of T is as follows $0 \to P_m \to \cdots \to P_1 \to P_0 \to T \to 0$. So, the complex

$$P^\bullet(T)\ :\ \cdots \to 0 \to P_m \to \cdots \to P_1 \to P_0 \to 0 \to \cdots$$

is a tilting complex in $\mathrm{D}\,(A)$ and also $B \simeq \mathrm{End}_{\mathrm{D}\,(A)}(P^\bullet(T))^{op}$. Thus, by 4.10 we get the result, since in this case n is $\mathrm{pd}\,T$. □

Remark 4.12. *Let A be an artin algebra. In* [2, Corollary 1]*, Bongartz consider a classical tilting module, that is, a tilting module $T \in \mathrm{mod}\,(A)$ such that $\mathrm{pd}\,T \leq 1$. The original Bongartz's result says that $\mathrm{gl.dim}\,(B) \leq \mathrm{gl.dim}\,(A) + 1$, where $B := \mathrm{End}_A(T)^{op}$.*

Finally, as another application of the main theorem, we get the following result for one-point extension algebras.

Corollary 4.13. *Let A and B be two finite-dimensional k-algebras, $M \in \mathrm{mod}\,(A)$ and $N \in \mathrm{mod}\,(B)$. Let $A[M]$ and $B[N]$ be the respective one-point extensions. If A and B are derived-equivalent, then the finiteness of the ϕ-dimension of one of the algebras $A, B, A[M]$ and $B[N]$ implies that all of them have finite ϕ-dimension.*

Proof. Let $X \in \mathrm{mod}\,(A[M])$. We know by Lemma 2.1 that $\phi(X) \leq 1 + \phi(\Omega X)$. Since ΩX can be seen as an A-modulo, it follows that $\phi\dim(A) < \infty$ implies $\phi\dim(A[M]) < \infty$. Now, the fact that $\mathrm{mod}\,(A) \subseteq \mathrm{mod}\,(A[M])$ gives us the other implication. From this fact and Theorem 4.10, we get that the following equivalences hold: $\phi\dim(A[M]) < \infty \Leftrightarrow \phi\dim(A) < \infty \Leftrightarrow \phi\dim(B) < \infty \Leftrightarrow \phi\dim(B[M]) < \infty$. □

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